Some Blow-Up Problems For A Semilinear Parabolic Equation With A Potential

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Abstract

The blow-up rate estimate for the solution to a semilinear parabolic equation $u_t = \Delta u + V(x)|u|^{p-1}u$ in $\Omega \times (0,T)$ with 0-Dirichlet boundary condition is obtained. As an application, it is shown that the asymptotic behavior of blow-up time and blow-up set of the problem with nonnegative initial data $u(x,0) = M\varphi(x)$ as M goes to infinity, which have been found in [5], are improved under some reasonable and weaker conditions compared with [5].

Key words: Blow-Up rate, Blow-Up time, Blow-Up set, Semilinear parabolic equations, Potential.

1 Introduction

In this paper, we are concerned with the following semilinear parabolic problem

$$\begin{cases} u_t = \Delta u + V(x)|u|^{p-1}u & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a bounded, convex, smooth domain, $1 < p < \frac{N+2}{N-2}$, $u_0 \in L^\infty(\Omega)$, and the potential $V \in C^1(\bar{\Omega})$ satisfies $V(x) \geqslant c$ for some positive constant c and all $x \in \Omega$. It is well-known that for any $u_0 \in L^\infty(\Omega)$ problem (1.1) has a unique local in time solution. Specially, if the L^∞-norm of the initial datum is small enough, then (1.1) has global, classical solution, while the solution to (1.1) ceases to exist after some time $T > 0$ and $\lim\limits_{t \uparrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$ provided that the initial datum u_0 is large in some suitable sense. In the latter case we call the solution u to (1.1) blows up in finite time and T the blow-up time. As usual, the blow-up set of the solution u is defined by

$$B[u] = \{x \in \bar{\Omega} \mid \text{there exist } x_n \to x, t_n \uparrow T, \text{such that } |u(x_n,t_n)| \to \infty\}.$$

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Much effort has been devoted to blow-up problems for semilinear parabolic equations since the pioneering works in 1960s due in particular to interest in understanding the mechanism of thermal runaway in combustion theory and as a model for reaction-diffusion. See for example, [1, 2, 3, 6, 7, 8, 14, 16]. The seminal works to problem (1.1) with $V(x) \equiv 1$ were done by Giga-Kohn [9, 10, 11]. In their paper [10], among other things, they have obtained a blow-up rate estimate, which is crucial to obtain the asymptotic behavior of the blow-up solution near the blow-up time. More precisely, under the assumptions that the domain Ω is the entire space or convex and the solution is nonnegative or $1 < p < \frac{3N+8}{3N-4}(N \geqslant 2)$ or $1 < p < \infty(N = 1)$, they proved that

$$|u(x,t)| \leqslant C(T-t)^{\frac{1}{p-1}}, \quad \forall\ (x,t) \in \Omega \times (0,T),$$

where $C > 0$ is a constant and $T > 0$ is the blow-up time. More recently, the same estimate has been obtained by Giga-Matsui-Sasayama [12, 13] for any subcritical p (i.e., $1 < p < \frac{N+2}{N-2}$ when $N \geqslant 3$, $1 < p < \infty$ when $N = 1, 2$).

Whether the similar blow-up rate estimate holds for the problem (1.1) for general potential V, to our best knowledge, is not well-understood up to now. Our first goal in this paper is to give an affirmative answer to this question. We have the following

Theorem 1.1. *Let u be a blow-up solution to* (1.1) *with a blow-up time T. There exists a positive constant C depending only on n, p, Ω, a bound for $T^{1/(p-1)}\|u_0\|_{L^\infty(\Omega)}$ and the positive lower bound c for V and $\|V\|_{C^1(\bar{\Omega})}$, such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C(T-t)^{-1/(p-1)}, \quad \forall\ t \in (0,T). \tag{1.2}$$

As in [10], we convert our problem to a uniform bound for a global in time solution w of the rescaled equation

$$w_s - \Delta w + \frac{1}{2} y \cdot \nabla w + \beta w - \bar{V}|w|^{p-1}w = 0, \quad \beta = \frac{1}{p-1},$$

with

$$w(y,s) = (T-t)^\beta u(a + y\sqrt{T-t}, t),\ \bar{V}(y,s) = V(a + ye^{-s/2}),$$

where $a \in \Omega$ is the center of the rescaling.

The proof of Theorem 1.1 depends heavily on the methods developed by Giga-Kohn in [10] and Giga-Matsui-Sasayama in [12, 13]. However our result is definitely not a direct consequence of their works. Due to the appearance of the potential V, some extra works should be done. It turns out that the key point and the main difference is to establish an upper bound for the global energy of w given by

$$E[w](s) = \frac{1}{2}\int_{\Omega(s)} (|\nabla w|^2 + \beta w^2)\rho\, dy - \frac{1}{p+1}\int_{\Omega(s)} \bar{V}|w|^{p+1}\rho\, dy,$$

where $\rho(y) = e^{-\frac{|y|^2}{4}}$. A lower bound for the energy can be obtained without much effort. When $V \equiv 1$, these bounds come easily from the Liapunov structure of the equation, i.e., the energy $E[w]$ is non-increasing in time. In our case this does not hold anymore. There is a "bad" term

$$\int_{\Omega(s)} \frac{\partial \bar{V}}{\partial s}|w|^{p+1}\rho\, dy$$

involved in the derivative of the energy $E[w]$. The main idea of proving Theorem 1.1 is as follows: First, using the fact that $\frac{\partial \bar{V}}{\partial s}$ is uniformly bounded in $\Omega(s)$ for all s, we get a rough control of the growth of the global energy $E[w]$. Since the term $\frac{\partial \bar{V}}{\partial s}$ can be written as $\nabla V(x) \cdot y e^{-s/2}$, we can use the information of the decay term $e^{-s/2}$. However, it has disadvantage that the unbounded thing y involves. Therefore we need some information from *higher level energies*

$$E_{2k}[w](s) = \frac{1}{2}\int_{\Omega(s)} (|\nabla w|^2 + \beta w^2)|y|^{2k}\rho\, dy - \frac{1}{p+1}\int_{\Omega(s)} \bar{V}|w|^{p+1}|y|^{2k}\rho\, dy, \quad k \in \mathbb{N}.$$

So our second step is to establish the control of the higher level energies. An upper bound for $E_{2k}[w]$ is obtained by an integral involving lower level energy. A lower bound for $E_{2k}[w]$ is obtained by two inequalities involving $\displaystyle\frac{d}{ds}\int_{\Omega(s)} w^2|y|^{2k}\rho\, dy$ and $dE_{2k}[w]/ds$. Finally we obtain an upper bound for the energy $E[w]$. To this end, the growth of lower level energies is improved by applying the growth of the higher level energies. An upper bound of the global energy $E[w]$ is obtained by a similar trick to bootstrap argument. Once these bounds are in hands, similar arguments to [12, 13] can be applied to show the boundedness of the global in time solution w, which in turn implies the blow-up rate estimate (1.2).

Another aim of this paper is to establish the asymptotic behavior of blow-up time and blow-up set of the blow-up solution to the problem (1.1) with nonnegative initial data $u_0 = M\varphi$ as $M \to \infty$. In this case, the problem we focused on can be rewritten as

$$\begin{cases} u_t = \Delta u + V(x)u^p & \text{in } \ \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = M\varphi(x) & \text{in } \ \Omega, \end{cases} \tag{1.3}$$

where $\varphi \in C(\bar{\Omega})$ satisfies $\varphi|_{\partial\Omega} = 0, \varphi(x) > 0, \ \forall\, x \in \Omega$ and V satisfies the same conditions as before. For these issues of blow-up problems to (1.3), we improve the results which have been obtained by Cortazar-Elgueta-Rossi [5] recently.

In [5], they have made some more technical condition on φ:

$$M\Delta\varphi + \frac{1}{2}\min_{x\in\Omega} V(x)M^p\varphi^p \geqslant 0. \tag{1.4}$$

The assumptions on Ω, p and V are the same as ours (although their assumption that V is Lipschitz is replaced by $V \in C^1(\bar{\Omega})$ in our case, our results still hold when V is Lipschitz). Under these assumptions, they proved that there exists $\bar{M} > 0$ such that if $M > \bar{M}$, then blow-up occurs and the blow-up time $T(M)$ and the blow-up set $B[u]$ of the blow-up solution to (1.3) satisfy

$$-\frac{C_1}{M^{\frac{p-1}{4}}} \leqslant T(M)M^{p-1} - \frac{A}{p-1} \leqslant \frac{C_2}{M^{\frac{p-1}{3}}},$$

$$\varphi^{p-1}(a)V(a) \geqslant \frac{1}{A} - \frac{C}{M^{\gamma}}, \quad \text{for all } \ a \in B[u],$$

where $A = \left(\max_{x\in\Omega}\varphi^{p-1}(x)V(x)\right)^{-1}$, $\gamma = \min(\frac{p-1}{4}, \frac{1}{3})$ and C_1, C_2 are two positive constants.

For the upper bound estimate on blow-up time, we have the following

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ be a smooth bounded domain, $p > 1$, V, φ be continuous functions on $\bar{\Omega}$ with $\varphi|_{\partial\Omega} = 0, \varphi(x) > 0, V(x) \geqslant c,\ \forall\ x \in \Omega$ for some $c > 0$. Then for any $k > p - 1$ there exists a constant $C > 0$ and $M_0 > 0$ such that for every $M \geqslant M_0$, the solution to* (1.3) *blows up in finite time that verifies*

$$T(M) \leqslant \frac{A}{(p-1)M^{p-1}} + CM^{-k}, \tag{1.5}$$

where $A = (\max_{x\in\Omega} \varphi^{p-1}(x)V(x))^{-1}$.

Remark 1.1. Our assumptions are weaker than ones in [5]. In [5], they required V and φ are Lipschitz continuous. Furthermore, our result tells that the decay of the upper bound of $T(M) - \frac{A}{(p-1)M^{p-1}}$ can be faster than which has been obtained in [5].

Notice that the proof of the upper bound of blow-up time in [5] depends on an argument of so-called "projection method" (see e.g. [14]) and the essential assumption that V, φ are Lipschitz continuous. Our proof of Theorem 1.2 requires a L^2-method (see e.g. [1]). The advantage of this method compared with one in [5] is that we do not need to control the first eigenvalue of Laplacian with Dirichlet boundary condition.

For the lower bound estimate for the blow-up time and the asymptotic behavior of blow-up set, we have

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ be a convex, bounded, smooth domain, $1 < p < \frac{N+2}{N-2}$, φ be a continuous function on $\bar{\Omega}$ with $\varphi|_{\partial\Omega} = 0, \varphi(x) > 0,\ \forall\ x \in \Omega$, and $V \in C^1(\bar{\Omega})$ with $V(x) > c,\ \forall\ x \in \Omega$ for some $c > 0$. Then there exist two positive constants C_1, C_2 such that*

$$T(M)M^{p-1} \geqslant -\frac{C_1}{M^{\frac{p-1}{4}}} \tag{1.6}$$

$$\varphi^{p-1}(a)V(a) \geqslant \frac{1}{A} - \frac{C_2}{M^{\frac{p-1}{4}}}, \quad \textit{for all} \ \ a \in B[u], \tag{1.7}$$

where $A = (\max_{x\in\Omega} \varphi^{p-1}(x)V(x))^{-1}$.

Applying Theorem 1.1 and the method in [5], we get Theorem 1.3 immediately. The only difference is that the role of Lemma 2.1 in [5] is replaced by that of our Theorem 1.1 now.

Remark 1.2. In our case, we do not need the assumption (1.4) anymore.

Remark 1.3. As described in [5], the asymptotics depend on a combination of the shape of both φ and V. To see this, if we drop the Laplacian, we get the ODE $u_t = V(x)u^p$ with initial condition $u(x, 0) = M\varphi(x)$. This gives $u(x, t) = C(T - t)^{-1/(p-1)}$ with

$$T = \frac{M^{1-p}}{(p-1)V(x)\varphi^{p-1}(x)}.$$

It turns out that blow-up occurs at point x_0 such that $V(x_0)\varphi^{p-1}(x_0) = \max_{x\in\Omega} V(x)\varphi^{p-1}(x)$. So the quantity $\max_{x\in\Omega} V(x)\varphi^{p-1}(x)$ plays a crucial role in the problem.

Remark 1.4. Also as in [5], (1.7) shows that the blow-up set concentrates when $M \to \infty$ near the set where $\varphi^{p-1}V$ attains its maximum. Notice that $1/A = \varphi^{p-1}(\bar{a})V(\bar{a})$ for any maximizer $\bar{a}$. If $\bar{a}$ is a non-degenerate maximizer, we conclude that there exist constants $c, d > 0$ such that

$$\varphi^{p-1}(\bar{a})V(\bar{a}) - \varphi^{p-1}(x)V(x) \geqslant c|\bar{a} - x|^2 \quad \text{for all} \quad x \in B(\bar{a}, d).$$

So (1.7) implies

$$|\bar{a} - a| \leqslant \frac{C}{M^{(p-1)/8}}, \quad \forall\, a \in B[u].$$

Throughout the paper we will denote by C a constant that does not depends on the solution itself. And it may change from line to line. And $K_1, K_2, \cdots,\ L_1, L_2, \cdots,$ $M_1, M_2, \cdots,\ N_1, N_2, \cdots,\ Q_1, Q_2, \cdots$ are positive constants depending on p, N, Ω, a lower bound of V, $\|V\|_{C^1(\bar{\Omega})}$ and the initial energy $E[w_0]$. Here and hereafter $w_0(y) = w(y, s_0)$.

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2 Blow-Up Rate Estimates

In this section, we will prove Theorem 1.1.

We introduce the rescaled function

$$w^a(y, s) = (T - t)^\beta u(a + y\sqrt{T - t}, t) \tag{2.1}$$

with $s = -\log(T - t)$, $\beta = \frac{1}{p-1}$. We shall denote w^a by w. If u solves (1.1), then w satisfies

$$w_s - \Delta w + \frac{1}{2} y \cdot \nabla w + \beta w - |w|^{p-1} w V(a + ye^{-s/2}) = 0 \quad \text{in } \Omega(s) \times (s_0, \infty) \tag{2.2}$$

where $\Omega(s) = \Omega_a(s) = \{y : a + ye^{-s/2} \in \Omega\}$, $s_0 = -\log T$.

We may assume $T = 1$ as in [12] so that we assume $s_0 = 0$. Here and hereafter we may denote $V(a + ye^{-s/2})$ by $\bar{V}(y, s)$.

By introducing a weight function $\rho(y) = \exp\left(-\frac{|y|^2}{4}\right)$, we can rewrite (2.2) as the divergence form:

$$\rho w_s = \nabla \cdot (\rho \nabla w) - \beta \rho w + \bar{V}|w|^{p-1} w \rho \quad \text{in } \Omega(s) \times (0, \infty). \tag{2.3}$$

As stated in [12], we may assume

$$w, w_s, \nabla w \text{ and } \nabla^2 w \text{ are bounded and continuous on } \Omega(s) \times [0, s] \text{ for all } s < \infty.$$

2.1 Global energy estimates

We introduce the energy of w of the form (we call it the "global energy")

$$E[w](s) = \frac{1}{2} \int_{\Omega(s)} (|\nabla w|^2 + \beta w^2) \rho\, dy - \frac{1}{p+1} \int_{\Omega(s)} \bar{V}|w|^{p+1} \rho\, dy.$$

We shall show that this global energy satisfies the following estimates.

Proposition 2.1. *Let w be a global solution of (2.3), then*

$$-K_1 \leqslant E[w] \leqslant K_2. \tag{2.4}$$

Proposition 2.2. *Let w be a global solution of (2.3), then*

$$\int_0^\infty \|w_s; L^2_\rho(\Omega(s))\|^2 ds \leqslant N_1, \tag{2.5}$$

$$\|w; L^2_\rho(\Omega(s))\|^2 \leqslant N_2, \tag{2.6}$$

$$\int_s^{s+1} \|w; L^{p+1}_\rho(\Omega(s))\|^{2(p+1)} ds \leqslant N_3. \tag{2.7}$$

We will prove these two properties in the following subsections.

2.1.1 Lower bound for $E[w]$

Lemma 2.3. $E[w] \geqslant -K_1$.

We see from (2.3) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)} w^2\rho\,dy &= \int_{\Omega(s)} ww_s\rho\,dy = \int_{\Omega(s)} w(\nabla\cdot(\rho\nabla w) - \beta\rho w + \bar V|w|^{p-1}w\rho)\,dy \\
&= -\int_{\Omega(s)} |\nabla w|^2\rho\,dy - \int_{\Omega(s)} \beta w^2\rho\,dy + \int_{\Omega(s)} \bar V|w|^{p+1}\rho\,dy \\
&= -2E[w] + \frac{p-1}{p+1}\int_{\Omega(s)} \bar V|w|^{p+1}\rho\,dy. \qquad (2.8)
\end{aligned}$$

Calculating the derivative of $E[w]$ and noting that $w_s|_{\partial\Omega(s)} = -\frac{1}{2}y\cdot\nabla w$ we have

$$\begin{aligned}
\frac{d}{ds}E[w](s) &= \int_{\Omega(s)} (\nabla w\cdot\nabla w_s + \beta ww_s)\rho\,dy - \int_{\Omega(s)} \bar V|w|^{p-1}ww_s\rho\,dy \\
&\quad + \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\gamma)\rho\,d\sigma - \frac{1}{p+1}\int_{\Omega(s)} \frac{\partial\bar V}{\partial s}|w|^{p+1}\rho\,dy \\
&= -\int_{\Omega(s)} \nabla\cdot(\rho\nabla w)w_s\,dy + \int_{\partial\Omega(s)} (\rho\nabla w\cdot\gamma)w_s\,d\sigma + \int_{\Omega(s)} \beta ww_s\rho\,dy \\
&\quad - \int_{\Omega(s)} \bar V|w|^{p-1}ww_s\rho\,dy + \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\gamma)\rho\,d\sigma - \frac{1}{p+1}\int_{\Omega(s)} \frac{\partial\bar V}{\partial s}|w|^{p+1}\rho\,dy \\
&= -\int_{\Omega(s)} \nabla\cdot(\rho\nabla w)w_s\,dy + \int_{\Omega(s)} \beta ww_s\rho\,dy - \int_{\Omega(s)} \bar V|w|^{p-1}ww_s\rho\,dy \\
&\quad - \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\gamma)\rho\,d\sigma + \frac{1}{2(p+1)}\int_{\Omega(s)} \nabla\bar V\cdot y|w|^{p+1}\rho\,dy \\
&= -\int_{\Omega(s)} w_s^2\rho\,dy - \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\gamma)\rho\,d\sigma \\
&\quad + \frac{1}{2(p+1)}\int_{\Omega(s)} \nabla\bar V\cdot y|w|^{p+1}\rho\,dy \qquad (2.9)
\end{aligned}$$

or

$$\begin{aligned}\int_{\Omega(s)} w_s^2 \rho\, dy &= -\frac{d}{ds}E[w](s) - \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2 (y\cdot\gamma)\rho\, d\sigma \\ &\quad + \frac{1}{2(p+1)}\int_{\Omega(s)} \nabla\bar{V}\cdot y|w|^{p+1}\rho\, dy. \end{aligned} \tag{2.10}$$

Notice that $\bar{V}$ is bounded. By (2.8), using Young's inequality, we have

$$\begin{aligned}-2E[w] + C\int_{\Omega(s)} |w|^{p+1}\rho\, dy &\leqslant -2E[w] + \frac{p-1}{p+1}\int_{\Omega(s)} \bar{V}|w|^{p+1}\rho\, dy \\ &= \int_{\Omega(s)} w w_s \rho\, dy \\ &\leqslant \varepsilon\int_{\Omega(s)} w_s^2\rho\, dy + \varepsilon\int_{\Omega(s)} |w|^{p+1}\rho\, dy + C(\varepsilon).\end{aligned}$$

Taking ε small enough we get

$$\int_{\Omega(s)} |w|^{p+1}\rho\, dy \leqslant CE[w] + \varepsilon\int_{\Omega(s)} w_s^2\rho\, dy + C(\varepsilon). \tag{2.11}$$

Since $\sup\limits_{y\in\Omega(s)} |\nabla\bar{V}||y| = \sup\limits_{x\in\Omega} |\nabla V||x-a|$ is bounded and Ω is convex, it follows from (2.9) and (2.11) that

$$\begin{aligned}\frac{d}{ds}E[w](s) &\leqslant -\int_{\Omega(s)} w_s^2\rho\, dy + C\int_{\Omega(s)} |w|^{p+1}\rho\, dy \\ &\leqslant -(1-\varepsilon)\int_{\Omega(s)} w_s^2\rho\, dy + CE[w] + C(\varepsilon).\end{aligned}$$

Take ε small then we have

$$\frac{d}{ds}E[w](s) \leqslant C_1 E[w] + C_2. \tag{2.12}$$

From this inequality, we claim that $E[w] \geqslant -\frac{C_2}{C_1}$. If not, then there exists $s_1 > 0$ such that $E[w](s_1) < -\frac{C_2}{C_1}$. By (2.12), we have $\frac{d}{ds}E[w](s_1) < 0$. This implies that

$$E[w](s) < -\frac{C_2}{C_1} \quad \text{for all } s \geqslant s_1.$$

Hence by (2.8) and Jensen's inequality, for $s \geqslant s_1$, we have

$$\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)} w^2\rho\, dy \geqslant C\int_{\Omega(s)} |w|^{p+1}\rho\, dy \geqslant C\left(\int_{\Omega(s)} w^2\rho\, dy\right)^{\frac{p+1}{2}}.$$

This fact shows that $\int_{\Omega(s)} w^2\rho\, dy$ will blow up in finite time, which is impossible.

2.1.2 Upper bound for $E[w]$

To find an upper bound for $E[w]$, we introduce

$$E_{2k}[w] = \frac{1}{2}\int_{\Omega(s)} (|\nabla w|^2 + \beta w^2)|y|^{2k}\rho\, dy - \frac{1}{p+1}\int_{\Omega(s)} \bar{V}|w|^{p+1}|y|^{2k}\rho\, dy, \quad k \in \mathbb{N}.$$

For this energy functional, we shall prove the following properties.

Proposition 2.4.

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\int_{\Omega(s)} w^2\rho|y|^{2k}\, dy &= -2E_{2k}[w] + \frac{p-1}{p+1}\int_{\Omega(s)} \bar{V}|w|^{p+1}\rho|y|^{2k}\, dy \\ &\quad + \int_{\Omega(s)} k\left(n+2k-2-\frac{1}{2}|y|^2\right) w^2|y|^{2k-2}\rho\, dy. \end{aligned} \tag{2.13}$$

Proposition 2.5.

$$\begin{aligned} \int_{\Omega(s)} w_s^2\rho|y|^{2k}\, dy &= -\frac{d}{ds}E_{2k}[w] - 2k\int_{\Omega(s)} \rho(y\cdot\nabla w)w_s|y|^{2k-2}\, dy - \frac{1}{4}\int_{\partial\Omega(s)} \left|\frac{\partial w}{\partial\gamma}\right|^2 (y\cdot\gamma)\rho|y|^{2k}\, d\sigma \\ &\quad -\frac{1}{p+1}\int_{\Omega(s)} \frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho|y|^{2k}\, dy. \end{aligned} \tag{2.14}$$

Proof of Proposition 2.4. Similar to that of [10] Proposition 4.1.
Proof of Proposition 2.5.

$$\begin{aligned} \frac{d}{ds}E_{2k}[w] &= \int_{\Omega(s)} (\nabla w\cdot\nabla w_s + \beta w w_s - \bar{V}|w|^{p-1}ww_s)\rho|y|^{2k}\, dy \\ &\quad -\frac{1}{p+1}\int_{\Omega(s)} \frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho|y|^{2k}\, dy + \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\gamma)\rho|y|^{2k}\, d\sigma. \end{aligned}$$

Estimating the first term of the right hand side, we get

$$\begin{aligned} \int_{\Omega(s)} \nabla w\cdot\nabla w_s\rho|y|^{2k}\, dy &= -\int_{\Omega(s)} \nabla\cdot(\rho|y|^{2k}\nabla w)w_s\, dy + \int_{\partial\Omega(s)} \rho|y|^{2k}\nabla w\cdot\gamma w_s\, d\sigma \\ &= -\int_{\Omega(s)} \nabla\cdot(\rho\nabla w)w_s|y|^{2k}\, dy - 2k\int_{\Omega(s)} w_s\rho\nabla w\cdot y|y|^{2k-2}\, dy \\ &\quad -\frac{1}{2}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\gamma)\rho|y|^{2k}\, d\sigma. \end{aligned}$$

Hence we have

$$\begin{aligned} \frac{d}{ds}E_{2k}[w] &= -\int_{\Omega(s)} w_s\left(\nabla\cdot(\rho\nabla w) + \beta w\rho - \bar{V}w^p\rho\right)|y|^{2k}\, dy - 2k\int_{\Omega(s)} w_s\rho\nabla w\cdot y|y|^{2k-2}\, dy \\ &\quad -\frac{1}{p+1}\int_{\Omega(s)} \frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho|y|^{2k}\, dy - \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\gamma)\rho|y|^{2k}\, d\sigma\, dy \\ &= -\int_{\Omega(s)} w_s^2\rho|y|^{2k}\, dy - 2k\int_{\Omega(s)} w_s\rho\nabla w\cdot y|y|^{2k-2}\, dy \\ &\quad -\frac{1}{p+1}\int_{\Omega(s)} \frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho|y|^{2k}\, dy - \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\gamma)\rho|y|^{2k}\, d\sigma. \end{aligned}$$

For $k=1$, similar to Proposition 4.2 of [10] we now state an parabolic type Pohozaev identity.

Proposition 2.6.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)}\left(\frac{1}{2}|y|^2-n\right)w^2\rho\,dy-(p+1)\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy\\
=\;&\int_{\Omega(s)}|\nabla w|^2\rho\left(c_2+\frac{p-1}{4}|y|^2\right)dy-\frac{p+1}{2}\int_{\partial\Omega(s)}\left|\frac{\partial w}{\partial\gamma}\right|^2(y\cdot\gamma)\rho\,d\sigma\\
&+\int_{\Omega(s)}\nabla\bar{V}\cdot y|w|^{p+1}\rho\,dy. \qquad (2.15)
\end{aligned}$$

We now define

$$\tilde{E}_2[w]\triangleq E_2[w]-\frac{1}{2}\int_{\Omega(s)}\left(\frac{1}{2}|y|^2-n\right)w^2\rho\,dy. \qquad (2.16)$$

Lemma 2.7.

$$\frac{d(\tilde{E}_2+c_3E)}{ds}\le -c_4\int_{\Omega(s)}(w_s^2+|\nabla w|^2)(1+|y|^2)\rho\,dy+\lambda(\tilde{E}_2+c_3E)+c_5, \qquad (2.17)$$

where $\lambda=\dfrac{8}{p-1}\dfrac{d_2}{d_1}$ *and* c_5 *depends on* p,d_1,d_2,η, d_1 *and* d_2 *are constants such that* $V(x)\geqslant d_1>0$ *and* $\sup\limits_{x\in\Omega}|\nabla V(x)|diam(\Omega)\leqslant 2d_2$ *and* η *is a small constant.*

Proof. By (2.14) and (2.15) we obtain that

$$\begin{aligned}
\frac{d\tilde{E}_2}{ds}=&-\int_{\Omega(s)}|w_s|^2\rho|y|^2\,dy-(p+3)\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy-\frac{1}{4}\int_{\partial\Omega(s)}(y\cdot\gamma)\left|\frac{\partial w}{\partial\gamma}\right|^2\rho|y|^2\,d\sigma\\
&-\int_{\Omega(s)}|\nabla w|^2\left(c_2+\frac{p-1}{4}|y|^2\right)\rho\,dy+\frac{p+1}{2}\int_{\partial\Omega(s)}(y\cdot\gamma)\left|\frac{\partial w}{\partial\gamma}\right|^2\rho\,d\sigma\\
&-\frac{1}{p+1}\int_{\Omega(s)}\frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho|y|^2\,dy+2\int_{\Omega(s)}\frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho\,dy. \qquad (2.18)
\end{aligned}$$

Since Ω is convex, the third term on the right is always negative. We control the second term by applying the Cauchy-Schwarz inequality: for any $\varepsilon>0$,

$$\left|\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy\right|\leqslant\varepsilon\int_{\Omega(s)}\rho|y|^2|\nabla w|^2\,dy+\frac{1}{4\varepsilon}\int_{\Omega(s)}\rho|w_s|^2\,dy.$$

Choosing ε small enough that $\frac{p-1}{4}-(p+3)\varepsilon=\delta>0$, we conclude that

$$\begin{aligned}
\frac{d\tilde{E}_2}{ds}\leqslant&-\int_{\Omega(s)}(|w_s|^2|y|^2+\delta|\nabla w|^2|y|^2+c_2|\nabla w|^2)\rho\,dy\\
&+\frac{p+1}{2}\int_{\partial\Omega(s)}(y\cdot\gamma)\left|\frac{\partial w}{\partial\gamma}\right|^2\rho\,d\sigma+\frac{p+3}{4\varepsilon}\int_{\Omega(s)}\rho|w_s|^2\,dy\\
&-\frac{1}{p+1}\int_{\Omega(s)}\frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho|y|^2\,dy+2\int_{\Omega(s)}\frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho\,dy.
\end{aligned}$$

Now choose $c_3 > \max(2(p+1), 1+\frac{p+3}{4\varepsilon})$, and apply (2.10) to get

$$\frac{p+1}{2}\int_{\partial\Omega(s)}(y\cdot\gamma)\left|\frac{\partial w}{\partial\gamma}\right|^2\rho\,d\sigma+\left(1+\frac{p+3}{4\varepsilon}\right)\int_{\Omega(s)}\rho|w_s|^2\,dy+c_3\frac{dE}{ds}\leqslant-\frac{c_3}{p+1}\int_{\Omega(s)}\frac{\partial\bar V}{\partial s}|w|^{p+1}\rho\,dy.$$

Let $2c_4 = \min(1,\delta,c_2) > 0$, we derive that

$$\begin{aligned}\frac{d(\tilde E_2+c_3E)}{ds} &\leq -2c_4\int_{\Omega(s)}(w_s^2+|\nabla w|^2)(1+|y|^2)\rho\,dy+2\int_{\Omega(s)}\frac{\partial\bar V}{\partial s}|w|^{p+1}\rho\,dy\\ &\quad -\frac{c_3}{p+1}\int_{\Omega(s)}\frac{\partial\bar V}{\partial s}|w|^{p+1}\rho\,dy-\frac{1}{p+1}\int_{\Omega(s)}\frac{\partial\bar V}{\partial s}|w|^{p+1}\rho|y|^2\,dy \qquad (2.19)\\ &\leqslant -2c_4\int_{\Omega(s)}(w_s^2+|\nabla w|^2)(1+|y|^2)\rho\,dy+\frac{2}{p+1}\int_{\Omega(s)}(c_3+|y|^2)\frac{\partial\bar V}{\partial s}|w|^{p+1}\rho\,dy.\end{aligned}$$

Note that $\bar V(y,s)\geqslant d_1>0$. From (2.8) we get

$$\frac{p-1}{p+1}d_1\int_{\Omega(s)}|w|^{p+1}\rho\,dy\leqslant\frac{p-1}{p+1}\int_{\Omega(s)}\bar V|w|^{p+1}\rho\,dy=2E[w]+\int_{\Omega(s)}ww_s\rho\,dy.$$

In the following we will denote $\frac{p+1}{(p-1)d_1}$ by $c(p,d_1)$. Making use of the inequality

$$ab\leqslant\varepsilon(a^2+b^{p+1})+C(\varepsilon),\quad p>1,\ \forall\,\varepsilon>0, \qquad (2.20)$$

we obtain that

$$\begin{aligned}\int_{\Omega(s)}|w|^{p+1}\rho\,dy &\leqslant 2c(p,d_1)E[w]+\int_{\Omega(s)}ww_sc(p,d_1)\rho\,dy\\ &\leqslant 2c(p,d_1)E[w]+\eta\int_{\Omega(s)}w^{p+1}\rho\,dy+\eta\int_{\Omega(s)}w_s^2\rho\,dy+C(p,d_1,\eta).\end{aligned}$$

Here and hereafter $C(p,d_1,\eta)$ denotes a constant depending on p,d_1,η and may be different at each occurrence. Take $\eta<1$ and we hence have

$$\int_{\Omega(s)}w^{p+1}\rho\,dy\leqslant\frac{2c(p,d_1)}{1-\eta}E[w]+\frac{\eta}{1-\eta}\int_{\Omega(s)}w_s^2\rho\,dy+C(p,d_1,\eta). \qquad (2.21)$$

From (2.13) we obtain that

$$\begin{aligned}\frac{p-1}{p+1}d_1\int_{\Omega(s)}|w|^{p+1}\rho|y|^2\,dy &\leqslant \frac{p-1}{p+1}\int_{\Omega(s)}\bar V|w|^{p+1}\rho|y|^2\,dy\\ &= 2E_2[w]+\int_{\Omega(s)}ww_s\rho|y|^2\,dy-\int_{\Omega(s)}\left(n-\frac12|y|^2\right)w^2\rho\,dy\\ &\leqslant 2\tilde E_2[w]+\int_{\Omega(s)}|ww_s||y|^2\rho\,dy+2\int_{\Omega(s)}\left(\frac12|y|^2-n\right)w^2\rho\,dy.\end{aligned}$$

Thanks to (2.20), we hence get

$$\begin{aligned}\int_{\Omega(s)}|w|^{p+1}\rho|y|^2\,dy &\leqslant 2c(p,d_1)\tilde E_2[w]+\int_{\Omega(s)}w^2|y|^{\frac{4}{p+1}}\cdot c(p,d_1)|y|^{\frac{2(p-1)}{p+1}}\cdot\rho\,dy\\ &\quad+\frac{\eta}{2}\int_{\Omega(s)}|w|^{p+1}\rho|y|^2\,dy+\frac{\eta}{2}\int_{\Omega(s)}w_s^2\rho|y|^2\,dy+C(p,d_1,\eta)\\ &\leqslant 2c(p,d_1)\tilde E_2[w]+\eta\int_{\Omega(s)}|w|^{p+1}\rho|y|^2\,dy+\frac{\eta}{2}\int_{\Omega(s)}w_s^2\rho|y|^2\,dy+C(p,d_1,\eta).\end{aligned}$$

Therefore we have

$$\int_{\Omega(s)} |w|^{p+1}\rho|y|^2\,dy \leqslant \frac{2c(p,d_1)}{1-\eta}\tilde{E}_2[w] + \frac{\eta}{2(1-\eta)}\int_{\Omega(s)} w_s^2\rho|y|^2\,dy + C(p,d_1,\eta). \qquad (2.22)$$

Combining (2.19) with (2.21) and (2.22) we obtain that

$$\begin{aligned}
\frac{d(\tilde{E}_2+c_3E)}{ds} \;&\leq\; -2c_4\int_{\Omega(s)}(|w_s|^2+|\nabla w|^2)(1+|y|^2)\rho\,dy + \frac{2}{p+1}c_3d_2\int_{\Omega(s)}|w|^{p+1}\rho\,dy \\
&\quad + \frac{2}{p+1}d_2\int_{\Omega(s)}|w|^{p+1}\rho|y|^2\,dy \\
&\leq\; \frac{2c(p,d_1)}{1-\eta}\frac{2}{p+1}c_3d_2E[w] + \left(\frac{2}{p+1}\frac{\eta}{1-\eta}c_3d_2 - 2c_4\right)\int_{\Omega(s)} w_s^2\rho\,dy \\
&\quad + C(p,d_1,d_2,\eta) + \frac{2c(p,d_1)}{1-\eta}\frac{2}{p+1}c_3d_2\tilde{E}_2[w] \\
&\quad + \left(\frac{2}{p+1}\frac{\eta}{2(1-\eta)}d_2 - 2c_4\right)\int_{\Omega(s)} w_s^2\rho|y|^2\,dy - c_4\int_{\Omega(s)}|\nabla w|^2(1+|y|^2)\rho\,dy,
\end{aligned}$$

where d_2 is a constant such that $\sup\left|\frac{\partial\bar{V}}{\partial s}\right| \leqslant d_2$. Take $\eta \leqslant \frac{1}{2}$ small enough such that $\frac{\eta d_2}{(p+1)(1-\eta)} \leqslant \frac{c_4}{c_3}$, then

$$\begin{aligned}
\frac{d(\tilde{E}_2+c_3E)}{ds} \;&\leq\; -c_4\int_{\Omega(s)}(|w_s|^2+|\nabla w|^2)(1+|y|^2)\rho\,dy + \frac{8}{(p-1)d_1}c_3d_2E[w] \\
&\quad + \frac{8}{(p-1)d_1}d_2\tilde{E}_2[w] + C(p,d_1,d_2,\eta).
\end{aligned}$$

Denote $\lambda = \frac{8}{p-1}\frac{d_2}{d_1}$, then we get

$$\frac{d(\tilde{E}_2+c_3E)}{ds} \leq -c_4\int_{\Omega(s)}(w_s^2+|\nabla w|^2)(1+|y|^2)\rho\,dy + \lambda(\tilde{E}_2+c_3E) + c_5,$$

where c_5 depends on p, d_1, d_2, η.

Lemma 2.8. *$\tilde{E}_2 + c_3E \geqslant -\bar{C}$, where $\bar{C}$ depends on p, d_1, d_2, η.*

Proof. From (2.13), using Jensen's inequality, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)} w^2\rho|y|^2\,dy \;&=\; -2\tilde{E}_2[w] + \frac{p-1}{p+1}\int_{\Omega(s)}\bar{V}|w|^{p+1}\rho|y|^2\,dy + 2\int_{\Omega(s)}\left(n - \frac{|y|^2}{2}\right)w^2\rho\,dy \\
&\geqslant\; -2\tilde{E}_2[w] - \int_{\Omega(s)} w^2\rho|y|^2\,dy + C\int_{\Omega(s)}|w|^{p+1}\rho|y|^2\,dy \\
&\geqslant\; -2\tilde{E}_2[w] + (C-\varepsilon)\int_{\Omega(s)}|w|^{p+1}\rho|y|^2\,dy - C(\varepsilon) \\
&\geqslant\; -2\tilde{E}_2[w] - C(\varepsilon) + C\left(\int_{\Omega(s)} w^2\rho|y|^2\,dy\right)^{\frac{p+1}{2}}.
\end{aligned}$$

This inequality plus $c_3\times$ (2.8) leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)} w^2\rho(|y|^2+c_3)\,dy \;&\geqslant\; -2c_3E[w]+c_3C\int_{\Omega(s)}|w|^{p+1}\rho\,dy-2\tilde{E}_2[w]\\ &\quad +C\left(\int_{\Omega(s)} w^2\rho|y|^2\,dy\right)^{\frac{p+1}{2}}-C(\varepsilon)\\ &\geqslant\; -2\left(\tilde{E}_2+c_3E+C(\varepsilon)\right)+C\left(\int_{\Omega(s)} w^2\rho(c_3+|y|^2)\,dy\right)^{\frac{p+1}{2}}.\end{aligned}$$

Denote $y(s)\triangleq\int_{\Omega(s)} w^2\rho(c_3+|y|^2)\,dy,\ J\triangleq\tilde{E}_2+c_3E,\ \bar{C}\triangleq\max\{C(\varepsilon),\dfrac{c_5}{\varepsilon}\}$. Then

$$\frac{1}{2}\frac{d}{ds}y(s)\geqslant -2(J+\bar{C})+Cy^{\frac{p+1}{2}}(s). \tag{2.23}$$

We claim that

$$J\geqslant -\bar{C}.$$

If not, there exists s_1 such that $J(s_1)<-\bar{C}$, then (2.17) tells us that

$$\left.\frac{d(J+\bar{C})}{ds}\right|_{s_1}\leqslant \varepsilon\left.\left(J+\frac{c_5}{\varepsilon}\right)\right|_{s_1}\leqslant \varepsilon(J+\bar{C})<0,$$

which shows that

$$J(s)<-\bar{C}\quad\forall\ s\geqslant s_1.$$

Therefore from (2.23) we get $\frac{1}{2}\frac{d}{ds}y(s)\geqslant Cy^{\frac{p+1}{2}}(s)$. From this inequality, we easily conclude that $y(s)$ will blow up in finite time, which is impossible. Hence our lemma holds.

To obtain rough estimates for the higher level energies, the following two inequalities, i.e. (2.26) and (2.27), play an important role. By Proposition 2.5 and Young's inequality, we have

$$\begin{aligned}\frac{d}{ds}E_{2k}[w] \;&=\; -\int_{\Omega(s)} w_s^2\rho|y|^{2k}\,dy-2k\int_{\Omega(s)}\rho(y\cdot\nabla w)w_s|y|^{2k-2}\,dy\\ &\quad -\frac{1}{4}\int_{\partial\Omega(s)}\left|\frac{\partial w}{\partial\gamma}\right|^2(y\cdot\gamma)\rho|y|^{2k}\,d\sigma-\frac{1}{p+1}\int_{\Omega(s)}\frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho|y|^{2k}\,dy\\ &\leqslant\; -(1-\varepsilon)\int_{\Omega(s)} w_s^2\rho|y|^{2k}\,dy+C(k,\varepsilon)\int_{\Omega(s)}|\nabla w|^2\rho|y|^{2k-2}\,dy\\ &\quad -\frac{1}{p+1}\int_{\Omega(s)}\frac{\partial\bar{V}}{\partial s}|w|^{p+1}\rho|y|^{2k}\,dy.\end{aligned} \tag{2.24}$$

Similar to (2.22), we have

$$\int_{\Omega(s)}|w|^{p+1}\rho|y|^{2k}\,dy\leqslant\frac{2c(p,d_1)}{1-\eta}E_{2k}[w]+\frac{\eta}{2(1-\eta)}\int_{\Omega(s)} w_s^2\rho|y|^{2k}\,dy+C(p,d_1,\eta). \tag{2.25}$$

Taking $\varepsilon,\eta>0$ small enough, we obtain that

$$\frac{d}{ds}E_{2k}[w]\leqslant -\frac{1}{2}\int_{\Omega(s)} w_s^2\rho|y|^{2k}\,dy+\mu E_{2k}[w]+C(\mu)+C(\mu)\int_{\Omega(s)}|\nabla w|^2\rho|y|^{2k-2}\,dy, \tag{2.26}$$

for all $\mu \geqslant \lambda$.

On the other hand, by Proposition 2.4, Hölder inequality, Young's inequality and Jensen's inequality we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)} w^2|y|^{2k}\rho\,dy &= -2E_{2k}[w] + \frac{p-1}{p+1}\int_{\Omega(s)} \bar{V}|w|^{p+1}\rho|y|^{2k}\,dy \\
&\quad + \int_{\Omega(s)} k\left(n+2k-2-\frac{1}{2}|y|^2\right) w^2|y|^{2k-2}\rho\,dy \\
&\geqslant -2E_{2k}[w] - C\int_{\Omega(s)} w^2|y|^{2k}\rho\,dy + C\int_{\Omega(s)} |w|^{p+1}|y|^{2k}\rho\,dy \\
&\geqslant -2E_{2k}[w] + (C-\varepsilon)\int_{\Omega(s)} |w|^{p+1}|y|^{2k}\rho\,dy - C(\varepsilon) \\
&\geqslant -2E_{2k}[w] - C + C\left(\int_{\Omega(s)} w^2|y|^{2k}\rho\,dy\right)^{\frac{p+1}{2}}.
\end{aligned} \tag{2.27}$$

Now we get following rough estimates

Lemma 2.9. *For any $k \in \mathbb{N}$, there exist positive constants L_k, M_k, N_k and Q_k, such that the following estimates hold:*

$$-L_k e^{2\lambda s} \leqslant E_{2k}[w](s) \leqslant M_k e^{2\lambda s},$$

$$\int_0^\infty e^{-2\lambda s}\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2k}\,dy\,ds \leqslant N_k,$$

$$\int_{\Omega(s)} w^2\rho|y|^{2k-2}\,dy \leqslant Q_k e^{2\lambda s}$$

for all $k \in \mathbb{N}$ and $s \geqslant 0$.

Proof. Let $\{\lambda_k\}_{k=1}^\infty \subset [\lambda, 2\lambda]$ be a strictly increasing sequence. It suffices to show the following estimates:

$$-L_k e^{\lambda_k s} \leqslant E_{2k}[w](s) \leqslant M_k e^{\lambda_k s}, \tag{2.28}$$

$$\int_0^\infty e^{-\lambda_k s}\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2k}\,dy\,ds \leqslant N_k, \tag{2.29}$$

$$\int_{\Omega(s)} w^2\rho|y|^{2k-2}\,dy \leqslant Q_k e^{\lambda_k s}. \tag{2.30}$$

We prove these estimates by induction.

Step 1. These estimates holds for $k = 1$.

Note that (2.17) gives us $\frac{d}{ds}\left(J + \frac{c_5}{\lambda}\right) \leqslant \lambda\left(J + \frac{c_5}{\lambda}\right)$, which imply that $J \leqslant Ce^{\lambda s}$. Therefore we now have $-\bar{C} \leqslant J \leqslant Ce^{\lambda s}$ by Lemma 2.8. Using the similar trick of getting (2.17), we can write (2.12) as a more refinement form:

$$\frac{d}{ds}\left(E[w] + \frac{c_2}{c_1}\right) \leqslant \lambda\left(E[w] + \frac{c_2}{c_1}\right),$$

then $E[w] \leqslant Ce^{\lambda s}$ and therefore $\tilde{E}_2[w] \geqslant -\bar{C} - c_3 E[w] \geqslant -Ce^{\lambda s}$. It follows that

$$|\tilde{E}_2[w]| \leqslant Ce^{\lambda s}. \tag{2.31}$$

From (2.17), we have $\frac{d}{ds}\left(J + \frac{c_5}{\lambda}\right) \leqslant -c_4 \int_{\Omega(s)} (w_s^2 + |\nabla w|^2)(1+|y|^2)\rho\, dy + \lambda\left(J + \frac{c_5}{\lambda}\right)$. Multiplying $e^{-\lambda s}$ on both sides and integrating from 0 to ∞, we obtain that

$$\int_0^\infty e^{-\lambda s} \int_{\Omega(s)} (w_s^2 + |\nabla w|^2)(1+|y|^2)\rho\, dyds \leqslant C. \tag{2.32}$$

In particular, (2.29) holds for $k = 1$.

Denote $y(s) = \int_{\Omega(s)} w^2 \rho\, dy$. Notice that

$$\begin{aligned}
\frac{d}{ds}\int_{\Omega(s)} w^2\rho\, dy &= -4E[w] + \frac{2(p-1)}{p+1}\int_{\Omega(s)} \bar{V}|w|^{p+1}\rho\, dy \\
&\geqslant -Ce^{\lambda s} + C\left(\int_{\Omega(s)} w^2\rho\, dy\right)^{\frac{p+1}{2}} \\
&= c_7\left(-c_8 e^{\lambda s} + \left(\int_{\Omega(s)} w^2\rho\, dy\right)^{\frac{p+1}{2}}\right).
\end{aligned}$$

If there exists $s_1 \geqslant 0$ such that $y(s_1) - 2c_8 e^{\lambda s_1} > 0$, then at s_1,

$$\begin{aligned}
\frac{d}{ds}(y(s) - 2c_8 e^{\lambda s})\Big|_{s_1} &= y'(s_1) - 2\lambda c_8 e^{\lambda s_1} \\
&\geqslant c_7\left(y(s_1)^{\frac{p+1}{2}} - c_8 e^{\lambda s_1}\right) - 2\lambda c_8 e^{\lambda s_1} \\
&= c_7\left(y(s_1)^{\frac{p+1}{2}} - c_8(1 + 2\lambda/c_7)e^{\lambda s_1}\right) \\
&> c_7\left(c_8^{\frac{p+1}{2}} e^{\frac{p+1}{2}\lambda s_1} - c_8(1+2\lambda/c_7)e^{\lambda s_1}\right) \\
&> 0,
\end{aligned}$$

since c_8 can be large enough. It follows that $y(s) > 2c_8 e^{\lambda s}$ for all $s > s_1$. So $y(s)^{\frac{p+1}{2}} > y(s) > 2c_8 e^{\lambda s}$ and then $\frac{d}{ds}y(s) \geqslant \frac{c_8}{2} y^{\frac{p+1}{2}}(s)$ for all $s > s_1$, which implies that y will blow up in finite time. This contradicts the fact that y is globally defined. So we have

$$y(s) \leqslant 2c_8 e^{\lambda s}, \quad \forall\ s \geqslant 0. \tag{2.33}$$

In other words, (2.30) holds for $k = 1$.

By (2.26),

$$\frac{d}{ds}\left(e^{-\lambda s}E_2[w]\right) \leqslant Ce^{-\lambda s}\int_{\Omega(s)} |\nabla w|^2 \rho\, dy + Ce^{-\lambda s}.$$

It follows from (2.32) that

$$E_2[w] \leqslant Ce^{\lambda s}.$$

On the other hand, by (2.31) and the definition of $\tilde{E}_2$, we have

$$\begin{aligned} -Ce^{\lambda s} &\leqslant \tilde{E}_2[w] = E_2[w] - \frac{1}{2}\int_{\Omega(s)}\left(\frac{1}{2}|y|^2 - n\right) w^2\rho\,dy \\ &\leqslant E_2[w] + \frac{n}{2}\int_{\Omega(s)} w^2\rho\,dy \\ &\leqslant E_2[w] + Ce^{\lambda s}, \end{aligned}$$

where the last inequality follows from (2.30) for $k = 1$. Therefore (2.28) also holds for $k = 1$.

Step 2. (2.28)-(2.30) holds for all $k \in \mathbb{N}$.

Suppose (2.28)-(2.30) holds for $k \leqslant n$. Since (2.28) holds for $k = n$, by (2.27) and a similar argument to derive (2.33) we conclude that (2.30) holds for $k = n+1$. By (2.26), we have

$$\frac{d}{ds}(e^{-\lambda_n s}E_{2n+2}[w]) \leqslant Ce^{-\lambda_n s}\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2n}\,dy + Ce^{-\lambda_n s}.$$

Since (2.29) holds for $k = n$, we have

$$e^{-\lambda_n s}E_{2n+2}[w] \leqslant C_n.$$

Now we need to obtain the lower bound for $E_{2n+2}[w]$. Denote

$$\begin{aligned} y(s) &= \int_{\Omega(s)} w^2\rho|y|^{2n+2}\,dy \\ z(s) &= E_{2n+2}[w] + C(\lambda_n). \end{aligned}$$

Then it follows from (2.26) and (2.27) that

$$y'(s) \geqslant -4z(s) + Cy^{\frac{p+1}{2}}(s) \tag{2.34}$$

$$z'(s) \leqslant \lambda_n z(s) + C\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2n}\,dy. \tag{2.35}$$

The last inequality implies that

$$\frac{d}{ds}(e^{-\lambda_n s}z(s)) \leqslant e^{-\lambda_n s}h(s), \tag{2.36}$$

where $h(s) = C\displaystyle\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2n}\,dy$. By induction hypothesis, we have

$$\int_0^\infty e^{-\lambda_n s}\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2n}\,dy \leqslant C_n. \tag{2.37}$$

We claim that

$$z(s) \geqslant -Ne^{\lambda_n s}, \quad \forall\ s \geqslant 0, \tag{2.38}$$

where $N = \int_0^\infty e^{-\lambda_n s}h(s)ds < \infty$.

Otherwise, there exists $s_1 \geqslant 0$ such that $e^{-\lambda_n s_1} z(s_1) + N < 0$. By (2.36), we have

$$e^{-\lambda_n s} z(s) - e^{-\lambda_n s_1} z(s_1) \leqslant \int_{s_1}^{s} e^{-\lambda_n \tau} h(\tau)\, d\tau \leqslant N,$$

for all $s > s_1$. So $e^{-\lambda_n s} z(s) \leqslant N + e^{-\lambda_n s_1} z(s_1) < 0$, i.e., $z(s) < 0$ for all $s > s_1$. Now from (2.34) we conclude that $y'(s) \geqslant C y^{\frac{p+1}{2}}(s)$ for all $s \geqslant s_1$, which implies $y(s)$ blows up in finite time. This is a contradiction. Therefore $E_{2n+2}[w] \geqslant -Ce^{\lambda_n s}$ and then $|E_{2n+2}[w]| \leqslant Ce^{\lambda_n s}$. In particular, (2.28) holds for $k = n+1$.

Finally, by (2.26), we have

$$\frac{d}{ds} E_{2n+2}[w] \leqslant -\frac{1}{2} \int_{\Omega(s)} w_s^2 \rho |y|^{2n+2}\, dy + C \int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n}\, dy + C + \lambda_n E_{2n+2}[w].$$

Combining this with the fact that $|E_{2n+2}[w]| \leqslant Ce^{\lambda_n s}$ and (2.37) we have

$$\int_0^\infty e^{-\lambda_n s} \int_{\Omega(s)} w_s^2 \rho |y|^{2n+2}\, dy\, ds \leqslant C.$$

By (2.25), we obtain

$$\begin{aligned}
\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n+2}\, dy &\leqslant 2E_{2n+2}[w] + \frac{2}{p+1} \int_{\Omega(s)} \bar{V} |w|^{p+1} \rho |y|^{2n+2}\, dy \\
&\leqslant CE_{2n+2}[w] + C + C \int_{\Omega(s)} w_s^2 \rho |y|^{2n+2}\, dy.
\end{aligned}$$

Therefore, by $|E_{2n+2}[w]| \leqslant Ce^{\lambda_n s}$, we get

$$\begin{aligned}
&\int_0^\infty e^{-\lambda_{n+1} s} |\nabla w|^2 \rho |y|^{2n+2}\, dy \\
\leqslant\ & C \int_0^\infty (E_{2n+2}[w] + 1) e^{-\lambda_{n+1} s} ds + C \int_0^\infty e^{-\lambda_n s} \int_{\Omega(s)} w_s^2 \rho |y|^{2n+2}\, dy\, ds \\
\leqslant\ & C \int_0^\infty e^{(\lambda_n - \lambda_{n+1}) s} ds + C \\
\leqslant\ & C.
\end{aligned}$$

Hence (2.29) holds for $k = n+1$. The Lemma is proved.

Remark 2.1. We have seen in the proof of this Lemma that

$$-L \leqslant E[w] \leqslant Ce^{\lambda s},$$

and

$$\int_0^\infty e^{-\lambda s} \int_{\Omega(s)} |\nabla w|^2 \rho\, dy\, ds \leqslant C.$$

Next, we need the following

Lemma 2.10. *Suppose $\lambda > \frac{1}{4}$ and for some $\alpha \in (\frac{1}{2}, 2\lambda]$, there exist positive constants M_k and N_k, such that*

$$|E_{2k}[w](s)| \leqslant M_k e^{\alpha s},$$

$$\int_0^\infty e^{-\alpha s} \int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2k} \, dy \, ds \leqslant N_k,$$

hold for all $k \in \mathbb{N} \cup \{0\}$ and $s \geqslant 0$. Then there exist positive constants M'_k and N'_k, such that

$$|E_{2k}[w](s)| \leqslant M'_k e^{(\alpha - \frac{1}{4})s},$$

$$\int_0^\infty e^{-(\alpha - \frac{1}{4})s} \int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2k} \, dy \, ds \leqslant N_k,$$

hold for all $k \in \mathbb{N} \cup \{0\}$ and $s \geqslant 0$. Here we set $E_0[w] = E[w]$.

Proof. Let $\{\delta_k\}_{k=0}^\infty \subset [\frac{1}{4}, \frac{1}{3}]$ be a strictly decreasing sequence. It suffices to show the following estimates:

$$|E_{2k}[w](s)| \leqslant M_k e^{(\alpha - \delta_k)s}, \tag{2.39}$$

$$\int_0^\infty e^{-(\alpha - \delta_k)s} \int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2k} \, dy \, ds \leqslant N_k. \tag{2.40}$$

We prove these estimates by induction.

Step 1. These estimates hold for $k = 0$.

Recalling (2.10) we have

$$\begin{aligned} \frac{dE}{ds} &\leqslant -\int_{\Omega(s)} w_s^2 \rho \, dy + \int_{\Omega(s)} \nabla V \cdot y e^{-s/2} |w|^{p+1} \rho \, dy \\ &\leqslant -\int_{\Omega(s)} w_s^2 \rho \, dy + C e^{-s/2} \int_{\Omega(s)} |y| |w|^{p+1} \rho \, dy \\ &\leqslant -\int_{\Omega(s)} w_s^2 \rho \, dy + C e^{-s/2} \int_{\Omega(s)} |y|^2 |w|^{p+1} \rho \, dy + C e^{-s/2} \int_{\Omega(s)} |w|^{p+1} \rho \, dy. \end{aligned} \tag{2.41}$$

Also we get

$$\begin{aligned} & e^{-s/2} \int_{\Omega(s)} |y|^2 |w|^{p+1} \rho \, dy \\ \leqslant\; & C e^{-s/2} \left(\int_{\Omega(s)} |\nabla w|^2 |y|^2 \rho \, dy + \int_{\Omega(s)} |w|^{p+1} \rho \, dy + C E_2[w] + C \right) \\ \leqslant\; & C e^{-s/2} \left(\int_{\Omega(s)} |\nabla w|^2 |y|^2 \rho \, dy + \int_{\Omega(s)} |w|^{p+1} \rho \, dy + C e^{\alpha s} + C \right). \end{aligned}$$

By (2.21) and the assumptions of this Lemma, then we get

$$\begin{aligned} \frac{d}{ds} E[w] &\leqslant -\frac{1}{2} \int_{\Omega(s)} w_s^2 \rho \, dy + C e^{-\frac{s}{2}} \int_{\Omega(s)} |\nabla w|^2 \rho |y|^2 \, dy + C e^{(\alpha - \frac{1}{2})s} + C e^{-\frac{1}{2}s} (E[w] + C) \\ &\leqslant -\frac{1}{2} \int_{\Omega(s)} w_s^2 \rho \, dy + C e^{-\frac{s}{2}} \int_{\Omega(s)} |\nabla w|^2 \rho |y|^2 \, dy + C e^{(\alpha - \frac{1}{2})s}. \end{aligned} \tag{2.42}$$

So

$$E[w](s) - E[w](0) \leqslant C \int_0^s e^{-\frac{\tau}{2}} \int_{\Omega(\tau)} |\nabla w|^2 \rho |y|^2 \, dy \, d\tau + Ce^{(\alpha-\frac{1}{2})s}.$$

We claim that

$$\int_0^s e^{-\frac{\tau}{2}} \int_{\Omega(\tau)} |\nabla w|^2 \rho |y|^2 \, dy \, d\tau \leqslant Ce^{(\alpha-\frac{1}{2})s}. \tag{2.43}$$

Indeed, if we denote the left hand side of (2.43) by $f(s)$, then $\int_0^\infty e^{-(\alpha-\frac{1}{2})s} f'(s) \, ds \leqslant C$ by the assumption. It follows that

$$C \geqslant \int_0^s e^{-(\alpha-\frac{1}{2})s} f'(s) \, ds \geqslant f(s) e^{-(\alpha-\frac{1}{2})s},$$

by integration by parts. So (2.43) holds and

$$E[w](s) \leqslant Ce^{(\alpha-\frac{1}{2})s}.$$

Notice that we have proved that $E[w] \geqslant -L$. Therefore (2.39) holds for $k = 0$.

By (2.42), (2.43) and $E[w] \geqslant -L$, we deduce that

$$\int_0^s \int_{\Omega(\tau)} w_s^2 \rho \, dy \, d\tau \leqslant Ce^{(\alpha-\frac{1}{2})s}. \tag{2.44}$$

As usual, we have

$$\begin{aligned} \int_{\Omega(s)} |\nabla w|^2 \rho \, dy &\leqslant 2E[w] + \frac{2}{p+1} \int_{\Omega(s)} \bar{V} |w|^{p+1} \rho \, dy \\ &\leqslant CE[w] + C \int_{\Omega(s)} w_s^2 \rho \, dy + C. \end{aligned}$$

Then

$$\begin{aligned} e^{-(\alpha-\frac{1}{3}s)} \int_{\Omega(s)} |\nabla w|^2 \rho \, dy &\leqslant C(E[w]+1) e^{-(\alpha-\frac{1}{3}s)} + Ce^{-(\alpha-\frac{1}{3}s)} \int_{\Omega(s)} w_s^2 \rho \, dy \\ &\leqslant Ce^{-\frac{1}{6}s} + Ce^{-(\alpha-\frac{1}{3}s)} \int_{\Omega(s)} w_s^2 \rho \, dy. \end{aligned}$$

Let $f(s) = \int_0^s \int_{\Omega(\tau)} w_s^2 \rho \, dy \, d\tau$. Then for any $s > 0$,

$$\begin{aligned} \int_0^s e^{-(\alpha-\frac{1}{3})\tau} \int_{\Omega(\tau)} w_s^2 \rho \, dy \, d\tau &= \int_0^s f'(\tau) e^{-(\alpha-\frac{1}{3})\tau} \, d\tau \\ &= f(s) e^{-(\alpha-\frac{1}{3})s} + (\alpha - \frac{1}{3}) \int_0^s f(\tau) e^{-(\alpha-\frac{1}{3})\tau} \, d\tau \\ &\leqslant C, \end{aligned}$$

due to (2.44). So

$$\begin{aligned} \int_0^\infty e^{-(\alpha-\frac{1}{3}\tau)} \int_{\Omega(\tau)} |\nabla w|^2 \rho \, dy \, d\tau &\leqslant C \int_0^\infty e^{-\frac{1}{6}\tau} \, d\tau + C \int_0^\infty e^{-(\alpha-\frac{1}{3}\tau)} \int_{\Omega(\tau)} w_s^2 \rho \, dy \, d\tau \\ &\leqslant C, \end{aligned}$$

i.e., (2.40) holds for $k = 0$.

Step 2. (2.39) and (2.40) hold for all $k \in \mathbb{N} \cup \{0\}$.

Suppose (2.39) and (2.40) hold for all $k = 0, 1, \cdots, n-1$. Taking $\varepsilon = 1/4$ in (2.24), we get

$$\begin{aligned}
\frac{dE_{2n}[w]}{ds} &\leqslant -\frac{3}{4}\int_{\Omega(s)} w_s^2 \rho |y|^{2n}\,dy + \frac{1}{p+1}\int_{\Omega(s)} \left|\frac{\partial \bar{V}}{\partial s}\right| |w|^{p+1}\rho|y|^{2n}\,dy + C\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n-2}\,dy \\
&\leqslant -\frac{3}{4}\int_{\Omega(s)} w_s^2 \rho |y|^{2n}\,dy + Ce^{-\frac{s}{2}}\int_{\Omega(s)} |w|^{p+1}\rho|y|^{2n+1}\,dy + C\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n-2}\,dy \\
&\leqslant -\frac{3}{4}\int_{\Omega(s)} w_s^2 \rho |y|^{2n}\,dy + C\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n-2}\,dy \\
&\quad + Ce^{-\frac{s}{2}}\left(\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n+2}\,dy + \int_{\Omega(s)} |w|^{p+1}\rho|y|^{2n}\,dy + C - CE_{2n+2}[w]\right) \\
&\leqslant -\frac{1}{2}\int_{\Omega(s)} w_s^2 \rho |y|^{2n}\,dy + C\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n-2}\,dy \\
&\quad + Ce^{-\frac{s}{2}}\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n+2}\,dy + Ce^{-\frac{s}{2}}(E_{2n}[w] + C) + Ce^{(\alpha-\frac{1}{2})s} \\
&\leqslant -\frac{1}{2}\int_{\Omega(s)} w_s^2 \rho |y|^{2n}\,dy + C\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n-2}\,dy \\
&\quad + Ce^{-\frac{s}{2}}\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n+2}\,dy + Ce^{(\alpha-\frac{1}{2})s}.
\end{aligned}$$

Notice that we have used that $\left|\frac{\partial \bar{V}}{\partial s}\right| \leqslant C|y|e^{-\frac{s}{2}}$ and the assumptions of the Lemma. Hence we get

$$\begin{aligned}
E_{2n}[w](s) - E_{2n}[w](0) &\leqslant C\int_0^s e^{-\frac{\tau}{2}}\int_{\Omega(\tau)} |\nabla w|^2 \rho |y|^{2n+2}\,dy\,d\tau + Ce^{(\alpha-\frac{1}{2})s} \\
&\quad + C\int_0^s e^{-\frac{\tau}{2}}\int_{\Omega(\tau)} |\nabla w|^2 \rho |y|^{2n-2}\,dy\,d\tau.
\end{aligned}$$

Since $\int_0^\infty e^{-\alpha s}\int_{\Omega(s)} |\nabla w|^2 \rho |y|^{2n+2}\,dy\,ds \leqslant N_{n+1}$, we get

$$\int_0^s e^{-\frac{\tau}{2}}\int_{\Omega(\tau)} |\nabla w|^2 \rho |y|^{2n+2}\,dy\,d\tau \leqslant Ce^{(\alpha-\frac{1}{2})s}$$

as before. Let $f(s) = \int_0^s \int_{\Omega(\tau)} |\nabla w|^2 \rho |y|^{2n-2}\,dy\,d\tau$. Then by induction hypothesis, we have

$$\int_0^\infty f'(s) e^{-(\alpha-\delta_{n-1})s}\,ds \leqslant N_{n-1}.$$

So

$$\begin{aligned}
\int_0^s f'(\tau)e^{-(\alpha-\delta_{n-1})\tau}\,d\tau &= f(s)e^{-(\alpha-\delta_{n-1})s} + (\alpha-\delta_{n-1})\int_0^s f(\tau)e^{-(\alpha-\delta_{n-1})\tau}\,d\tau \\
&\geqslant f(s)e^{-(\alpha-\delta_{n-1})s},
\end{aligned}$$

i.e., $f(s) \leqslant N_{n-1}e^{(\alpha-\delta_{n-1})s}$.

Therefore

$$E_{2n}[w] \leqslant N_n e^{(\alpha-\delta_{n-1})s}. \tag{2.45}$$

Now let $y(s) = \int_{\Omega(s)} w^2\rho|y|^{2n}\,dy$, $z(s) = E_{2n}[w] + C$. Then by (2.26) and (2.27), we have

$$\begin{aligned} y'(s) &\geqslant -4z(s) + Cy^{\frac{p+1}{2}}(s), \\ z'(s) &\leqslant 2\lambda z(s) + C\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2n-2}\,dy \triangleq 2\lambda z(s) + h(s). \end{aligned}$$

Since $\alpha < 2\lambda$, $z'(s) \leqslant (\alpha - \delta_n')z(s) + g(s)$, where $g(s) = (2\lambda - \alpha + \delta_n')z(s) + h(s)$ and $\delta_n' \in (\delta_n, \delta_{n-1})$. It follows from (2.45) and induction hypothesis that

$$\begin{aligned} \int_0^\infty e^{-(\alpha-\delta_n')s} g(s)\,ds &\leqslant C\int_0^\infty e^{(\delta_n'-\delta_{n-1})s}\,ds + C\int_0^\infty e^{-(\alpha-\delta_n')s}\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2n-2}\,dy\,ds \\ &\leqslant C. \end{aligned}$$

A similar argument to obtain (2.38) gives us

$$z(s) \geqslant -Ce^{(\alpha-\delta_n')s}. \tag{2.46}$$

From (2.45) and (2.46), we know that (2.39) holds for $k = n$.

From the fact that

$$\frac{dE_{2n}[w]}{ds} \leqslant -\frac{1}{2}\int_{\Omega(s)} w_s^2\rho|y|^{2n}\,dy + (\alpha - \delta_n')E_{2n}[w] + g(s) + C$$

and above estimates, we have

$$\int_0^\infty e^{-(\alpha-\delta_n')s}\int_{\Omega(s)} w_s^2\rho|y|^{2n}\,dy\,ds \leqslant C.$$

As before, we have

$$\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2n}\,dy \leqslant CE_{2n}[w] + C\int_{\Omega(s)} w_s^2\rho|y|^{2n}\,dy + C.$$

Multiplying $e^{-(\alpha-\delta_n)s}$ on both sides and integrating over $(0,\infty)$, we obtain

$$\begin{aligned} &\int_0^\infty e^{-(\alpha-\delta_n)s}\int_{\Omega(s)} |\nabla w|^2\rho|y|^{2n}\,dy\,ds \\ \leqslant\ & C\int_0^\infty e^{-(\alpha-\delta_n)s}e^{(\alpha-\delta_n')s}\,ds + C + C\int_0^\infty e^{-(\alpha-\delta_n')s}\int_{\Omega(s)} w_s^2\rho|y|^{2n}\,dy\,ds \\ \leqslant\ & C, \end{aligned}$$

i.e., (2.40) holds for $k = n$. So the proof of this Lemma is complete.

To obtain the upper bound of $E[w]$, we also need the following

Lemma 2.11. *Suppose that there exist two positive constants M, N and some $\alpha \in (0, \frac{1}{2})$ such that*

$$|E_2[w](s)| \leqslant Me^{\alpha s},$$
$$\int_0^\infty e^{-\alpha s} \int_{\Omega(s)} |\nabla w|^2 \rho |y|^2 \, dy \, ds \leqslant N.$$

Then we have

$$E[w] \leqslant K_2.$$

Proof. Recall from (2.41) that

$$\frac{dE}{ds} \leqslant -\int_{\Omega(s)} w_s^2 \rho \, dy + Ce^{-s/2} \int_{\Omega(s)} |y|^2 |w|^{p+1} \rho \, dy + Ce^{-s/2} \int_{\Omega(s)} |w|^{p+1} \rho \, dy.$$

By the lower bound of E_2 and Young's inequality, we get

$$\begin{aligned} e^{-s/2} \int_{\Omega(s)} |y|^2 |w|^{p+1} \rho \, dy &\leqslant Ce^{-s/2} \left(\int_{\Omega(s)} |\nabla w|^2 |y|^2 \rho \, dy + \int_{\Omega(s)} |w|^{p+1} \rho \, dy + Ce^{\alpha s} + C \right) \\ &\leqslant Ce^{-s/2} \int_{\Omega(s)} |\nabla w|^2 |y|^2 \rho \, dy + Ce^{-s/2} \int_{\Omega(s)} |w|^{p+1} \rho \, dy \\ &\quad + Ce^{-s/2} + Ce^{(\alpha - \frac{1}{2})s}. \end{aligned} \tag{2.47}$$

Using (2.11), we have

$$\begin{aligned} \frac{dE}{ds} &\leqslant -\int_{\Omega(s)} w_s^2 \rho \, dy + Ce^{-s/2} \int_{\Omega(s)} |\nabla w|^2 |y|^2 \rho \, dy \\ &\quad + Ce^{-s/2} \int_{\Omega(s)} |w|^{p+1} \rho \, dy + Ce^{-s/2} + Ce^{(\alpha - \frac{1}{2})s} \\ &\leqslant -\frac{1}{2} \int_{\Omega(s)} w_s^2 \rho \, dy + Ce^{-s/2} \int_{\Omega(s)} |\nabla w|^2 |y|^2 \rho \, dy \\ &\quad + Ce^{-s/2}(E[w] + C) + Ce^{(\alpha - \frac{1}{2})s}. \end{aligned} \tag{2.48}$$

By Lemma 2.3, we may assume $E[w] + C > 1$. So

$$\frac{d}{ds} \log(E[w] + C) \leqslant Ce^{-s/2} \int_{\Omega(s)} |\nabla w|^2 |y|^2 \rho \, dy + Ce^{-s/2} + Ce^{(\alpha - \frac{1}{2})s}.$$

Noticing that $\alpha < \frac{1}{2}$, we obtain that $E[w] \leqslant K_2$ from the assumptions.

Proof of Proposition 2.1 Combining Lemma 2.11 with Lemma 2.9, Lemma 2.10 and Remark 2.1, we get the upper bound of $E[w]$ immediately. Notice that the lower bound of $E[w]$ has been obtained in Lemma 2.3. So the proof is complete.

2.1.3 Proof of Proposition 2.2

Proof of (2.5)**.** From (2.11) we have

$$\int_{\Omega(s)} |w|^{p+1} \rho \, dy \leqslant \varepsilon \int_{\Omega(s)} w_s^2 \rho \, dy + C(\varepsilon).$$

Then (2.48) tells us that

$$\frac{dE}{ds} \leqslant \left(-\frac{1}{2} - \varepsilon e^{-s/2}\right) \int_{\Omega(s)} w_s^2 \rho\, dy + C(\varepsilon) e^{-s/2} + f(s),$$

where $f(s) = Ce^{-s/2} \int_{\Omega(s)} |\nabla w|^2 |y|^2 \rho\, dy$, which is an integrable function. Integrating this inequality from s_0 to T, we get

$$\frac{1}{4} \int_{s_0}^{T}\int_{\Omega(s)} w_s^2 \rho\, dy \leqslant \int_{s_0}^{T} \left(Ce^{-s/2} + f(s)\right) ds + E(s_0) - E(T).$$

It follows that

$$\int_0^\infty \|w_s; L^2_\rho(\Omega(s))\|^2 ds \leqslant N_1.$$

Proof of (2.6)**.** Making use of Jensen's inequality, from (2.8), we get

$$\frac{1}{2}\frac{d}{ds} \int_{\Omega(s)} w^2 \rho\, dy \geqslant -2K_2 + C(p, d_2, \Omega) \left(\int_{\Omega(s)} w^2 \rho\, dy\right)^{\frac{p+1}{2}}.$$

We assert that

$$\int_{\Omega(s)} w^2 \rho\, dy \leqslant N_2,$$

where $N_2 = \left(\dfrac{2K_2}{C(p, d_2, \Omega)}\right)^{\frac{2}{p+1}}$ is the zero of $-2K_2 + C(p, d_2, \Omega) x^{\frac{p+1}{2}} = 0$.

If not, there exists s_1 such that

$$\int_{\Omega(s_1)} w^2 \rho\, dy > \left(\frac{2K_2}{C(p, d_2, \Omega)}\right)^{\frac{2}{p+1}}.$$

Then

$$\frac{1}{2}\frac{d}{ds} \int_{\Omega(s)} w^2 \rho\, dy \bigg|_{s=s_1} > C > 0,$$

which implies that

$$\int_{\Omega(s)} w^2 \rho\, dy > 2C \quad \forall\ s > s_1.$$

Then there exists some $\bar{t}$ such that for $s > \bar{t}$,

$$-2K_2 + C(p, d_2, \Omega) \left(\int_{\Omega(s)} w^2 \rho\, dy\right)^{\frac{p+1}{2}} \geqslant \frac{C(p, d_2, \Omega)}{2} \left(\int_{\Omega(s)} w^2 \rho\, dy\right)^{\frac{p+1}{2}}$$

so that y blows up in finite time, which is impossible.

Proof of (2.7)**.** Recall that $\bar{V} \geqslant d_1$ and $E[w] \leqslant K_2$. Then from (2.8) we see that

$$\int_{\Omega(s)} |w|^{p+1} \rho\, dy \leqslant \varepsilon \frac{2(p+1)}{d_1(p-1)} K_2 + \frac{p+1}{d_1(p-1)} \left(\int_{\Omega(s)} |w|^2 \rho\, dy\right)^{\frac{1}{2}} \left(\int_{\Omega(s)} |w_s|^2 \rho\, dy\right)^{\frac{1}{2}}.$$

Therefore by (2.5) and (2.6) we have

$$\int_s^{s+1} \left(\int_{\Omega(s)} |w|^{p+1} \rho\, dy\right)^2 ds \leqslant C + CN_2 \int_0^\infty \int_{\Omega(s)} |w_s|^2 \rho\, dy \leqslant N_3.$$

2.2 Proof of Theorem 1.1

Let $\psi \in C^2(\mathbb{R}^n)$ be a bounded function with supp$\psi \subset B_{2R}(0) \cap \Omega$. Then ψw satisfies

$$\rho(\psi w)_s - \nabla \cdot (\rho \nabla(\psi w)) + \nabla \cdot (\rho w \nabla \psi) + \rho \nabla \psi \cdot \nabla w + \beta \psi \rho w - \bar{V} \psi |w|^{p-1} w \rho = 0 \quad \text{in } \Omega(s) \times (0, \infty). \tag{2.49}$$

We introduce two types of local energy.

$$E_\psi[w](s) = \frac{1}{2} \int_{\Omega(s)} \left(|\nabla(\psi w)|^2 + (\beta \psi^2 - \nabla |\psi|^2) w^2 \right) \rho \, dy - \frac{1}{p+1} \int_{\Omega(s)} \bar{V} \psi^2 |w|^{p+1} \rho \, dy, \tag{2.50}$$

$$\mathcal{E}_\psi[w](s) = \frac{1}{2} \int_{\Omega(s)} \psi^2 (|\nabla w|^2 + \beta w^2) \rho \, dy - \frac{1}{p+1} \int_{\Omega(s)} \bar{V} \psi^2 |w|^{p+1} \rho \, dy. \tag{2.51}$$

By the similar trick of [12], we could establish a lower and an upper bound for $\mathcal{E}_\psi[w]$. We just list some important results and ignore the proof.

2.2.1 Upper bound for $\mathcal{E}_\psi[w]$

Using (2.4) and (2.6) we obtain that

$$\|w(s); W^{1,2}_\rho(\Omega(s))\|^2 \leqslant K_1(1 + \|w_s(s); L^2_\rho(\Omega(s))\|) \quad \text{for all } s \geqslant 0, \tag{2.52}$$

where $\|w(s); W^{1,2}_\rho(\Omega(s))\|^2 = \beta \|w(s); L^2_\rho(\Omega(s))\|^2 + \|\nabla w(s); L^2_\rho(\Omega(s))\|^2$.

Proposition 2.12. *(Quasi-monotonicity of $\mathcal{E}_\psi[w]$)*

$$\frac{d}{ds} \mathcal{E}_\psi[w](s) \leqslant L_1(1 + \|w_s(s); L^2_\rho(\Omega(s))\|) + Ce^{-s/2} \int_{\Omega(s)} \psi^2 |y||w|^{p+1} \rho \, dy \tag{2.53}$$

for all $s > 0$.

Proposition 2.13. *There exists a positive constant K_2, such that*

$$\int_s^{s+1} \mathcal{E}_\psi[w](\tau) \, d\tau \leqslant K_2 \quad \text{for all } s \geqslant 0, \tag{2.54}$$

where K_2 depends on $n, p, \|\psi\|_\infty$, upper bound for $\mathcal{E}_\psi[w]$ and upper bound for $\bar{V}$.

Note that

$$\int_s^{s+1} Ce^{-\tau/2} \int_{\Omega(\tau)} \psi^2 |y||w|^{p+1} \rho \, dy d\tau \leqslant C.$$

Thanks to (2.53), (2.5) and (2.54) we can derive an upper bound for $\mathcal{E}_\psi[w]$.

Theorem 2.14.

$$\mathcal{E}_\psi[w] \leqslant M \quad \text{for all } s \geqslant 0. \tag{2.55}$$

2.2.2 Lower bound for $\mathcal{E}_\psi[w]$

Notice that

$$E_\psi - \mathcal{E}_\psi = \int_{\Omega(s)} \psi w (\nabla\psi \cdot \nabla w)\rho\, dy.$$

By estimating $|E_\psi - \mathcal{E}_\psi|$ and using (2.6) we obtain

Proposition 2.15. *There exists a positive constant J_1 such that*

$$\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)} |\psi w|^2 \rho\, dy \geqslant -2\mathcal{E}_\psi - J_1 + \frac{p-1}{p+1}\int_{\Omega(s)} \bar{V}\psi^2 |w|^{p+1}\rho |y|^2\, dy. \tag{2.56}$$

By (2.56), (2.53) and (2.5) we obtain that

Theorem 2.16. *There exists a positive constant L_2 such that*

$$\mathcal{E}_\psi[w](s) \geqslant -L_2 \quad \textit{for all } s \geqslant 0. \tag{2.57}$$

Once we have these bounds for the local energies, the proof of Theorem 1.1 follows from bootstrap arguments, an interpolation theorem in [4] and the interior regular theorem in [15] as in [12, 13]. We omit the details since there is no anything new.

Remark 2.2. If we only treat nonnegative solution to (1.1), then Theorem 1.1 can be proved through the bounds we have obtained in Section 2.1. We can combine the methods in [10] and [17] to get the blow-up rate estimate.

3 Asymptotic behavior of the Blow-Up Time and Blow-Up set

In this section, we are interested in the following problem

$$\begin{cases} u_t = \Delta u + V(x)u^p & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = M\varphi(x) & \text{in } \Omega, \end{cases}$$

where $\varphi \in C(\bar{\Omega})$ satisfies $\varphi|_{\partial\Omega} = 0, \varphi(x) > 0,\ \forall\, x \in \Omega$ and V satisfies the conditions described as in Section 1.

The main goal of this section is to prove Theorem 1.2 and 1.3.

Proof of Theorem 1.2. That blow-up occurs for large M is standard fact. Let $\bar{a} \in \Omega$ such that $\varphi^{p-1}(\bar{a})V(\bar{a}) = \max_x \varphi^{p-1}(x)V(x)$.

Since φ and V are continuous, it follows that $\forall\, \varepsilon > 0, \exists\, \delta > 0$, such that

$$V(x) > V(\bar{a}) - \frac{\varepsilon}{2},\quad \varphi(x) > \varphi(\bar{a}) - \frac{\varepsilon}{2},\quad \forall\, x \in B(\bar{a},\delta).$$

Let w be the solution of

$$\begin{cases} w_t = \Delta w + \left(V(\bar{a}) - \frac{\varepsilon}{2}\right) w^p & \text{in } B(\bar{a},\delta) \times (0,T_w), \\ w = 0 & \text{on } \partial B(\bar{a},\delta) \times (0,T_w), \\ w(x,0) = M\left(\varphi(\bar{a}) - \varepsilon\right) & \text{in } B(\bar{a},\delta) \end{cases} \tag{3.1}$$

and T_w its corresponding blow up time.

A comparison argument shows that $u \geq w$ in $B(\bar{a},\delta)\times(0,T)$ and hence $T \leq T_w$.

Our goal is to estimate T_w for large values of M. Define

$$I(w)=\frac{1}{2}\int_{B(\bar{a},\delta)}|\nabla w|^2\,dx-\frac{V(\bar{a})-\dfrac{\varepsilon}{2}}{p+1}\int_{B(\bar{a},\delta)}w^{p+1}\,dx,$$

then

$$\begin{aligned}
I'(t) &= \int_{B(\bar{a},\delta)}\nabla w\cdot\nabla w_t\,dx-\left(V(\bar{a})-\frac{\varepsilon}{2}\right)\int_{B(\bar{a},\delta)}w^p w_t\,dx\\
&= -\int_{B(\bar{a},\delta)}w_t\left(\Delta w+\left(V(\bar{a})-\frac{\varepsilon}{2}\right)w^p\right)dx\\
&= -\int_{B(\bar{a},\delta)}w_t^2\,dx.
\end{aligned}$$

Set $\Phi(t)=\dfrac{1}{2}\displaystyle\int_{B(\bar{a},\delta)}w^2(x,t)\,dx$, then we obtain that

$$\begin{aligned}
\Phi'(t) &= \int_{B(\bar{a},\delta)}ww_t\,dx\\
&= \int_{B(\bar{a},\delta)}w\left(\Delta w+\left(V(\bar{a})-\frac{\varepsilon}{2}\right)w^p\right)dx\\
&= -\int_{B(\bar{a},\delta)}|\nabla w|^2+\left(V(\bar{a})-\frac{\varepsilon}{2}\right)\int_{B(\bar{a},\delta)}w^{p+1}\,dx\\
&= -2I(w)+\frac{p-1}{p+1}\left(V(\bar{a})-\frac{\varepsilon}{2}\right)\int_{B(\bar{a},\delta)}w^{p+1}\,dx\\
&> -2I(w)+\frac{p-1}{p+1}\left(V(\bar{a})-\varepsilon\right)|B|^{\frac{1-p}{2}}\left(\int_{B(\bar{a},\delta)}w^2\,dx\right)^{\frac{1+p}{2}}\\
&= -2I(w_0)+2\int_0^t\int_{B(\bar{a},\delta)}w_t^2\,dxdt+\tilde{C}\Phi^{\frac{1+p}{2}}(t), \qquad (3.2)
\end{aligned}$$

where $\tilde{C}=\dfrac{p-1}{p+1}\left(V(\bar{a})-\varepsilon\right)|B|^{\frac{1-p}{2}}2^{\frac{1-p}{2}}$.

In particular, $\Phi'(t)>0$.

On the other hand,

$$\Phi'(t)=\int_{B(\bar{a},\delta)}ww_t\,dx\leqslant\left(\int_{B(\bar{a},\delta)}w^2\,dx\right)^{\frac{1}{2}}\left(\int_{B(\bar{a},\delta)}w_t^2\,dx\right)^{\frac{1}{2}}=(2\Phi(t))^{\frac{1}{2}}\left(\int_{B(\bar{a},\delta)}w_t^2\,dx\right)^{\frac{1}{2}},$$

which tells us that $\displaystyle\int_{B(\bar{a},\delta)}w_t^2\,dx\geqslant\frac{(\Phi'(t))^2}{2\Phi(t)}$. Therefore from (3.2) we get

$$\Phi'(t)>-2I(w_0)+\int_0^t\frac{(\Phi'(t))^2}{\Phi(t)}\,dt+\tilde{C}\Phi^{\frac{1+p}{2}}(t).$$

Set $f(t) = -2I(w_0) + \int_0^t \frac{(\Phi'(t))^2}{\Phi(t)}\,dt$ and $g(t) = \frac{2}{p-1}\tilde{C}\Phi^{\frac{1+p}{2}}(t)$.

Note that

$$f(0) = -2I(w_0) = \frac{2}{p+1}\left(V(\bar{a}) - \frac{\varepsilon}{2}\right)|B|M^{p+1}(\varphi(\bar{a}) - \varepsilon)^{p+1},$$

$$g(0) = \frac{2}{p+1}(V(\bar{a}) - \varepsilon)|B|M^{p+1}(\varphi(\bar{a}) - \varepsilon)^{p+1}.$$

It follows that $f(0) > g(0)$. Hence

$$\Phi'(0) > f(0) + \tilde{C}\Phi^{\frac{1+p}{2}}(0) > g(0) + \tilde{C}\Phi^{\frac{1+p}{2}}(0) = \frac{p+1}{p-1}\tilde{C}\Phi^{\frac{1+p}{2}}(0).$$

Then $\exists\, \eta > 0$, such that $\Phi'(t) \geqslant \frac{p+1}{p-1}\tilde{C}\Phi^{\frac{1+p}{2}}(t), \quad t \in [0, \eta]$.

Define $A = \{\theta \in [0, T_\Phi] : \Phi'(t) \geqslant \frac{p+1}{p-1}\tilde{C}\Phi^{\frac{1+p}{2}}(t), \quad t \in [0, \theta]\}$, where T_Φ is the blow-up time of Φ. Then A is closed. On the other hand, A is open. In fact, $\forall\, \theta \in A$, since

$$f'(t) = \frac{(\Phi'(t))^2}{\Phi(t)}, \quad g'(t) = \frac{p+1}{p-1}\tilde{C}\Phi^{\frac{p-1}{2}}(t)\Phi'(t),$$

it follows that $f'(t) > g'(t)$ for $t \in [0, \theta]$.

Recall that $f(0) > g(0)$. We conclude that

$$f(t) > g(t), \quad t \in [0, \theta].$$

In particular, $f(\theta) > g(\theta)$.

Thus, there exists $\bar{\beta} > 0$ such that for all $\beta \in [0, \bar{\beta}]$, $f(\theta + \beta) > g(\theta + \beta)$ or

$$\Phi'(\theta + \beta) > \frac{p+1}{p-1}\tilde{C}\Phi^{\frac{1+p}{2}}(\theta + \beta),$$

which means $\theta + \bar{\beta} \in A$. Therefore $A = [0, T_\Phi]$. In other words,

$$\Phi'(t) \geqslant \frac{p+1}{p-1}\tilde{C}\Phi^{\frac{1+p}{2}}(t), \quad t \in [0, T_\Phi].$$

Integrating this inequality from 0 to T_Φ, we get

$$T_\Phi \leqslant \frac{1}{(p-1)(V(\bar{a}) - \varepsilon)M^{p-1}(\varphi(\bar{a}) - \varepsilon)^{p-1}}.$$

Since $\varepsilon > 0$ is arbitrarily small, the Theorem follows readily from the above estimate.

Proof of Theorem 1.3. The proof is almost the same as in [5]. The only different thing is that we improve their Lemma 2.2. For the reader's convenience, we outline the proof here.

Let M be large such that the solution u blows up in finite time $T = T(M)$ and let $a = a(M)$ be a blow-up point. To involve the information of T, we modify the definition of w to be

$$w(y, s) = (T - t)^{\frac{1}{p-1}} u(a + y(T - t)^{\frac{1}{2}}, t)|_{t = T(1 - e^{-s})}.$$

Then w satisfies

$$\rho w_s = \nabla \cdot (\rho \nabla w) - \beta \rho w + V(a + yT^{\frac{1}{2}} e^{-\frac{s}{2}})|w|^{p-1} w\rho \quad \text{in } \Omega(s) \times (0, \infty),$$

where $\Omega(s) = \{y | a + yT^{\frac{1}{2}} e^{-\frac{s}{2}} \in \Omega\}$.

Consider the frozen energy

$$E(w) = \int_{\Omega(s)} \left(\frac{1}{2}|\nabla w|^2 + \frac{\beta}{2} w^2 - \frac{1}{p+1} V(a) w^{p+1} \right) \rho \, dy.$$

Then

$$\begin{aligned} \frac{dE}{ds} &\leqslant -\int_{\Omega(s)} w_s^2 \rho \, dy + \int_{\Omega(s)} (V(a + yT^{\frac{1}{2}} e^{-\frac{s}{2}}) - V(a)) w^p w_s \rho \, dy \\ &\leqslant -\int_{\Omega(s)} w_s^2 \rho \, dy + CT^{\frac{1}{2}} e^{-\frac{s}{2}} \left(\int_{\Omega(s)} w_s^2 \rho \, dy \right)^{\frac{1}{2}}. \end{aligned}$$

We have used Theorem 1.1 and Hölder inequality in the last inequality. So $\frac{dE}{ds} \leqslant CTe^{-s}$, and then

$$E(w) \leqslant E(w_0) + CT.$$

Since w is bounded, by the argument of [10] and [11], we conclude that

$$\lim_{s \to \infty} w(y, s) = k(a) \triangleq \frac{1}{((p-1)V(a))^{\frac{1}{p-1}}}$$

uniformly in any compact set, and

$$E(w(\cdot, s)) \to E(k(a)) \quad \text{as } s \to \infty.$$

So

$$E(k(a)) \leqslant E(w_0) + CT. \tag{3.3}$$

By Theorem 1.2, we estimate $E(w_0)$ to get $E(w_0) \leqslant E(T^{\frac{1}{p-1}} M\varphi(a)) + CT^{\frac{1}{2}}$. So

$$E(k(a)) \leqslant E(T^{\frac{1}{p-1}} M\varphi(a)) + CT^{\frac{1}{2}}$$

Observe that $E(b) = \Gamma F(b)$ for any constant b, where $\Gamma = \int \rho \, dy$ and $F(x) = \frac{1}{2\beta} x^2 - \frac{1}{p+1} V(a) x^{p+1}$. It follows that F attains a unique maximum at $k(a)$ and there exist α, β such that if $|x - k(a)| < \alpha$ then $F''(x) < -1/2$ and if $|F(x) - F(k(a))| < \beta$ then $|x - k(a)| < \alpha$. From (3.3), we have $F(k(a)) \leqslant F(T^{\frac{1}{p-1}} M\varphi(a)) + CT^{\frac{1}{2}}$. By the properties of F we have

$$CT^{\frac{1}{2}} \geqslant F(k(a)) - F(T^{\frac{1}{p-1}} M\varphi(a)) \geqslant \frac{1}{4}(k(a) - T^{\frac{1}{p-1}} M\varphi(a))^2.$$

By Theorem 1.2, for any $k > 0$ there exists $M_k > 0$ such that if $M > M_k$, we have

$$\begin{aligned} k(a) - CT^{\frac{1}{4}} &\leqslant T^{\frac{1}{p-1}} M\varphi(a) \\ &\leqslant k(a)\theta(a) + \frac{C\varphi(a)}{M^k}, \end{aligned}$$

where

$$\theta(a)=\frac{\varphi(a)V(a)^{\frac{1}{p-1}}}{\varphi(\bar a)V(\bar a)^{\frac{1}{p-1}}},\quad \varphi(\bar a)V(\bar a)^{\frac{1}{p-1}}=\max_{x\in\Omega}\varphi(x)V(x)^{\frac{1}{p-1}}.$$

Therefore, we get

$$k(a)(1-\theta(a))\leqslant\frac{C\varphi(a)}{M^k}+\frac{C}{M^{\frac{p-1}{4}}}\leqslant\frac{C}{M^{\frac{p-1}{4}}}$$

if we choose $k>\frac{p-1}{4}$. Then

$$\theta(a)\geqslant 1-\frac{C}{M^{\frac{p-1}{4}}}.$$

This implies

$$\varphi(a)V(a)^{\frac{1}{p-1}}\geqslant\varphi(\bar a)V(\bar a)^{\frac{1}{p-1}}-\frac{C}{M^{\frac{p-1}{4}}}.$$

We can deduce from this inequality that $\varphi(a)\geqslant C>0$ for large M. So

$$\frac{1}{\varphi(a)((p-1)V(a))^{\frac{1}{p-1}}}-\frac{CT^{\frac14}}{\varphi(a)}\leqslant MT^{\frac{1}{p-1}}.$$

Therefore

$$\frac{1}{\varphi(\bar a)((p-1)V(\bar a))^{\frac{1}{p-1}}}-CT^{\frac14}\leqslant MT^{\frac{1}{p-1}},$$

i.e.,

$$\frac{1}{\varphi(\bar a)((p-1)V(\bar a))^{\frac{1}{p-1}}}-\frac{C}{M^{\frac{1}{p-1}}}\leqslant MT^{\frac{1}{p-1}}.$$

The Theorem is proved.

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